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Radiative Heat Transfer in the Presence of Obscurations

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I. Introduction

THE radiative heat transfer problem, in axisymmetric geometry and in the presence of an active gas, frequently occurs in the thermal design of aircrafts and spacecrafts, and as such it was considered by many authors (see, e.g., Refs. 1-7). In a nonparticipating medium, an important part of this calculation reduces to the shape factor. For complicated and/or multilayered axisymmetric bodies the approaches mentioned are of limited practical value, since the problem must then be solved numerically. This requires a subdivision of the whole body into a large number of surface elements, and the calculation of the radiative heat exchange between each pair of them. One is immediately faced with the complication introduced by the presence of full or partial obscurations produced by the various parts of the body. Though it is often the most time-consuming part of the whole calculation, it did not seem to have received a systematic treatment. The present Note aims to close this gap. It gives a short account of a conceptually different approach for the resolution of obscurations, and summarizes the corresponding algorithm. The method was tested, and it was utilized in a comprehensive computer program. Further details on the algorithm and on its incorporation in a general radiative heat transfer scheme is found in our report.[8]

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II. Obscurations in Axisymmetric Bodies

We start by analyzing the elementary situation shown in Fig. 1. The circles $S(x_\ell)$ and $S(x_m)$ are sections of a radiating surface at $x=x_\ell$ and $x=x_m$, with radii R_ℓ and R_m, respectively. The ring $S(x)$ is a cross section at x of an opaque layer formed by revolution of two profiles around the x axis; $r_i(x)$ is the running radius of the inner profile, while $r_e(x)$ is the running radius of the exterior profile. Consider the line of sight between a point P_m of $S(x_m)$, and P_ℓ, located at the bottom of $S(x_\ell)$. Let P be the point of intersection of $P_\ell P_m$ with the plane of $S(x)$, and let R be the distance of P to the axis. Denote by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ three mutually orthogonal coordinate unit vectors; and by $\boldsymbol{u}$, $\boldsymbol{u}_m$ and ω, unit vectors, pointing respectively in the $\boldsymbol{xP}$, $\boldsymbol{x_m P_m}$, and $\boldsymbol{P_\ell P}$ directions. Let further q and q_m be the lengths of $P_\ell P$ and of $P_\ell P_m$. The vision between P_ℓ and P_m is obstructed by $S(x)$ when P is located within the bounds of $S(x)$, that is, when R satisfies

$$r_i(x) < R < r_e(x) \tag{1a}$$

This will be transformed by the use of the following vector equations, deducible from Fig. 1,

$$\boldsymbol{i}(x-x_\ell) + \boldsymbol{u}R = -\boldsymbol{j}R_\ell + \omega q$$

$$\boldsymbol{i}(x_m-x_\ell) + \boldsymbol{u}_m R_m = -\boldsymbol{j}R_\ell + \omega q_m$$

$$\boldsymbol{u}_m = \boldsymbol{k}\cos[\phi - (\pi/2)] + \boldsymbol{j}\cos(\pi-\phi)$$

After some algebra we arrive at

$$\mu_{-1}(x) < \mu < \mu_1(x) \tag{1b}$$

where

$$\mu = \cos\phi, \quad \mu_{-1}(x) = \max\{-1,\ C[r_i(x)]\}$$

$$\mu_1(x) = \min\{1, C[r_e(x)]\}$$

$$C[r(x)] = \frac{[(x_m-x_\ell)r(x)]^2 - [(x_m-x)R_\ell]^2 - [(x-x_\ell)R_m]^2}{2[(x_m-x)R_\ell][(x-x_\ell)R_m]} \tag{2}$$

and $r(x)$ is either $r_i(x)$ or $r_e(x)$. Angles ϕ, satisfying both inequalities, correspond to lines of sight $P_\ell P_m$, obscured by the section $S(x)$.

The interval $[\mu_{-1}(x),\ \mu_1(x)]$ will be referred to as the interval of obscuration between $S(x_\ell)$ and $S(x_m)$, produced by the section $S(x)$. We shall extend the preceding argument to an opaque, smoothly changing layer occupying an interval $a \le x \le b$, between x_ℓ and x_m. Since the ring $S(x)$ is a section of the layer at x, it is evident that the domain of obscuration between $S(x_\ell)$ and $S(x_m)$, produced by the whole layer, is equal to the union of intervals $\bigcup_{a\le x\le b}[\mu_{-1}(x),\ \mu_1(x)]$. As x sweeps out $[a, b]$, the corresponding intervals of obscuration $[\mu_{-1}(x),\ \mu_1(x)]$, change continuously. It follows that the whole union is a single interval, say (μ_{-1}, μ_1). Thus,

$$\mu_{-1} = \min_{a\le x\le b} \mu_{-1}(x), \quad \mu_1 = \max_{a\le x\le b} \mu_1(x) \tag{3}$$

are the limits of the interval of obscuration produced by the whole layer.

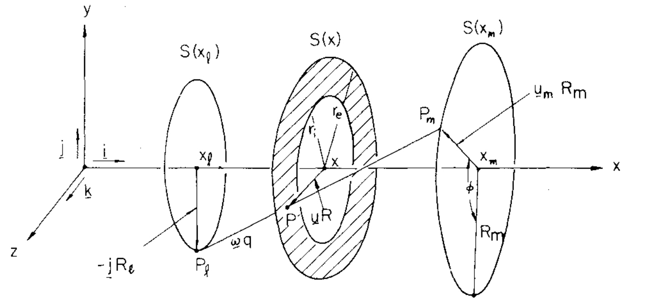

Fig. 1 Obscuration of the line of sight $P_\ell P_m$ produced by the ring $S(x)$.

Two special cases are of frequent occurrence.

1) *An envelope* enclosing the radiating system. It has the same effect on the transport of radiation as a layer with an infinite exterior radius $[r_e(x) = +\infty]$. It follows from Eqs. (2) and (3) that $\mu_I = 1$ in this case.

2) *An inner body* of the system. It has the same effect as a layer with $r_i(x) = 0$. Equations (2) and (3) imply now $\mu_{-I} = -1$.

We turn to the general case of a multilayered axisymmetric body separating two circles $S(x_\ell)$ and $S(x_m)$. The domain of obscuration produced by the whole body is equal to the union of obscuration intervals (μ^k_{-I}, μ^k_I) produced by its constituent envelope, layers, and innerbodies. This domain will generally consist of several separate intervals. The domain of vision (complement to the domain of obscuration) may therefore also consist of several separate intervals. However, if proper layers (both r^k_i and r^k_e are $\neq 0$) are absent, the domain of vision reduces to a single interval, equal to the intersection of the intervals of vision produced by its envelope and innerbodies. It is evident that an efficient scheme of calculation should enable a fast evaluation and an orderly classification of the various extrema of the functions $\mu^k_{-I}(x)$, $\mu^k_I(x)$. It turned out that, unless one restricts himself to piecewise-linear profiles, numerous particular cases have to be considered and the schemes of computation become impractical. On the other hand, an approximation of arbitrary profiles by piecewise linear functions results in a simple and fast algorithm. We shall assume therefore that $r^k_e(x)$ and $r^k_i(x)$ are piecewise-linear functions of x. We refer to any conical piece as a "zone." The kth zone is called internal if it is formed by some $r^k_i(x)$, and external if it is formed by some $r^k_e(x)$. The internal zones determine μ^k_{-I}, and the external zones determine μ^k_I. The explicit formulas for μ_{-I} and μ_I will now be determined.

III. Technical Considerations

Let $r(x)$ be the current radius of a zone (internal or external) that may obstruct the vision between $S(x_\ell)$ an $S(x_m)$. We shall restrict the zone to its "obscuring part," that is, the part located between x_ℓ and x_m. Let the corresponding restricted domain be $a \leq x \leq b$ (at this stage of analysis we limit ourselves to $x_\ell < a < b < x_m$). As $r(x)$ is linear,

$$r(x) = \frac{q_\ell}{x_m - x_\ell}(x_m - x) + \frac{q_m}{x_m - x_\ell}(x - x_\ell) \tag{4}$$

where q_ℓ and q_m are constants. With the notation

$$z = \frac{R_m}{R_\ell}\,\frac{x - x_\ell}{x_m - x}, \quad p_\ell = q_\ell / R_\ell, \quad p_m = q_m / R_m \tag{5}$$

Eq. (2) becomes

$$2C = (p_m^2 - 1)z + (p_\ell^2 - 1)z^{-1} + 2p_m p_\ell \tag{6}$$

For $a < x < b$, $dz/dx > 0$. The correspondence between x and z is thus one-to-one, and z can be used as a free variable instead of x. The domain of z is $[z_a, z_b]$, where $z_a \geq \epsilon > 0$ and $z_b \leq 1/\epsilon < \infty$, for some sufficiently small ϵ. Now, $dC/dz = 0$ and $z > 0$ imply that for $(p_\ell^2 - 1)(p_m^2 - 1) \leq 0$, C is a monotonic function of z or a constant. The extremal values are thus reached at the ends z_a and z_b, of the relevant interval of z. By Eq. (5),

$$z_a = \frac{R_m}{R_\ell}\,\frac{a - x_\ell}{x_m - a}, \qquad z_b = \frac{R_m}{R_\ell}\,\frac{b - x_\ell}{x_m - b} \tag{7}$$

On the other hand, for $(p_m^2 - 1)(p_\ell^2 - 1) > 0$, the equation $dC/dz = 0$ has a unique positive solution

$$z^* = \sqrt{(p_\ell^2 - 1)/(p_m^2 - 1)} \tag{8a}$$

and

$$C(z^*) = \operatorname{sgn}(p^2 - 1)\sqrt{(p_\ell^2 - 1)(p_m^2 - 1)} + p_\ell p_m \tag{8b}$$

$C(z^*)$ is a maximum for $p_\ell^2 - 1 < 0$, and a minimum for $p_\ell^2 - 1 > 0$. The extremum provided by Eqs. (8) is not always the relevant one. First, the condition $z_a \leq z^* \leq z_b$ must be satisfied, to ensure that the critically obscuring section lies within the boundaries of the zone. Second, if we are considering an internal zone, and $p_\ell^2 - 1 < 0$, the extremum is irrelevant, since it is a maximum, while we are looking for a minimum. Finally, if we are considering an external zone, and $p_\ell^2 - 1 > 0$, the extremum is again irrelevant, since it is a minimum, while we are looking for a maximum. In all these disabling cases, the desired solution is either $C(z_a)$ or $C(z_b)$. A simple scheme of computation of μ_{-I} and μ_I can now be stated. It would, however, be incomplete as it would be unable to treat the ubiquitous cases $x_\ell = a$, $x_m = b$ or $x_\ell = x_m$. In addition, it would suffer from a kind of an instability which originates from a genuine physical situation, and therefore cannot be eliminated by an algebraic manipulation. Consider, for instance, the case in which both circles S_ℓ and S_m lie on the same internal zone. If the radius of one of the circles is increased by any amount, the obscuration by the common zone would be complete; on the other hand, if the radius is decreased, there will be no obscuration at all. This can be formalized as follows: if the value of $A = p_\ell^2 - 1$ is close to zero, then small variations of A (even such that are due to round-off errors) can induce significant variations in the obscuration interval. The same kind of argument is true for $B = p_m^2 - 1$. This proposition requires, for verification, the consideration of a number of different geometrical configurations. We shall skip it for brevity. In order to eliminate the round-off error in the case A or $B = 0$, we adopt the following procedure. Whenever the value of A or $B < \epsilon$, we set the corresponding value exactly to zero. The small quantity ϵ should be chosen to be slightly larger than the maximal possible round-off error, and much smaller than the accuracy of the data. According to the preceding discussion, we compute $C(z)$ not from Eq. (6), but from the "equivalent" expression

$$C(z) = \tfrac{1}{2}[Bz + (A/z)] + p_\ell p_m \tag{9}$$

We omit for brevity the discussion of the remaining exceptional cases and proceed with the general algorithm. We introduce the notion of the zone parameter J, which is equal to -1 for an internal zone, and $+1$ for an external zone. Before executing the algorithm, the small quantity ϵ, the zone parameters p_ℓ, p_m, a, b, z_a, z_b, J, and the coordinates of the circles x_ℓ and x_m have to be defined.

IV. The Algorithm

1) Compute $A = p_\ell^2 - 1$; $B = p_m^2 - 1$.
2) If $|A| < \epsilon$ then $A = 0$.
3) If $|B| < \epsilon$ then $B = 0$.
4) If $x_\ell = x_m$ then $\mu_{-J} = -1$ and $\mu_J = (AB)^{1/2} + p_\ell p_m$.
5) If $x_\ell \neq x_m$ then
   a) replace a by $a + \epsilon$, and b by $b - \epsilon$.
   b) if $B \neq 0$ then $z^* = |A/B|^{1/2}$, else $z^* = 0$.
   c) If $JA < 0$ and $JB < 0$ and $z_a \leq z^* \leq z_b$ then $\mu_J' = -J\cdot(AB)^{1/2} + p_\ell p_m$ else $\mu_J' = J\cdot\max[J\cdot C(z_a), J\cdot C(z_b)]$.
   d) Set $\mu_J = J\cdot\min[1, J\cdot\mu_J']$; $\mu_{-J} = -J$.

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Growth and Decay of Weak Waves in Radiative Magnetogasdynamics

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Introduction

Because of the high ambient temperatures that prevail in many phenomena, it is of interest to consider the effects of thermal radiation in gasdynamics. Since at high temperatures a gas is likely to be at least partially ionized, electromagnetic effects can also be significant. One is thus led to study the interaction of the radiation and electromagnetic effects that may arise in problems of solar photosphere, rocket re-entry, and elsewhere. This Note extends the work of Singh and Sharma[1] to the magnetogasdynamic regime, and the effects of radiative transfer are treated by the use of a differential approximation which is valid over the entire optical depth range from the transparent limit (considered in Ref. 1) to the optically thick limit. The magnetic field is taken to be transverse to the direction of propagation of the wave.

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Basic Equations

The fundamental equations are

$$(\partial\rho/\partial t) + u_i\rho_{,i} + \rho u_{i,i} = 0 \tag{1}$$

$$\rho(\partial u_i/\partial t) + \rho u_j u_{i,j} + (p+p_R)_{,i} + \mu H_j(H_{j,i} - H_{i,j}) = 0 \tag{2}$$

$$(\partial H_i/\partial t) + u_j H_{i,j} - H_j u_{i,j} + H_i u_{j,j} = 0 \tag{3}$$

$$\rho\left\{\partial\left(\frac{p}{\rho(\gamma-1)} + \frac{E_R}{\rho}\right)\Big/\partial t\right\} + \rho u_i\left(\frac{p}{\rho(\gamma-1)} + \frac{E_R}{\rho}\right)_{,i} + (p+p_R)u_{i,i} + q_{i,i} = 0 \tag{4}$$

$$p = \rho \mathcal{R} T \tag{5}$$

where q_i ($i=1,2,3$) and H_i are, respectively, the components of the radiative flux vector and the magnetic field strength, μ the magnetic permeability, and $\mathcal{R}$ the gas constant. The remaining symbols have the same definitions as in Ref. 1. The Maxwell equation $H_{i,i}$ is essentially included in Eq. (3) because the divergence of that equation gives $\partial(H_{i,i})/\partial t = 0$.

Within the differential approximation the equations of radiative transfer[2] may be written as the pair

$$(\partial E_R/\partial t) + q_{i,i} = -\bar{k}_p(cE_R - 4\sigma T^4) \tag{6}$$

$$(1/c)(\partial q_i/\partial t) + (c/3)E_{R,i} = -\bar{k}_p q_i \tag{7}$$

where $\bar{k}_p$ is the absorption coefficient depending on ρ and T, σ Stefan's constant, and c the velocity of light. Further, within the order of this approximation, the radiative pressure p_R is one-third of the radiative energy E_R.

Equations (1) and (4) yield

$$(\partial p/\partial t) + u_i p_{,i} + \rho a^2 u_{i,i} + 3(\gamma-1)\{(\partial p_R/\partial t) + u_i p_{R,i}\} + (\gamma-1)q_{i,i} = 0 \tag{8}$$

where $a = \{(\gamma p + 4(\gamma-1)p_R)/\rho\}^{1/2}$ is the effective acoustic speed.

A moving singular surface Σ, across which the flow parameters are essentially continuous but the discontinuities in their derivatives are permitted, is called a *weak wave*. Forming jumps across Σ in Eqs. (1-3) and (6-8) and using $u_n = 0$ and the compatibility conditions,[3] we get

$$G\zeta = \rho_0\lambda_i n_i \tag{9}$$

$$\rho_0 G\lambda_i = \xi n_i + \theta n_i + \mu H_j\eta_j n_i \tag{10}$$

$$G\eta_i = H_i\lambda_j n_j \tag{11}$$

$$3G\theta = \epsilon_i n_i \tag{12}$$

$$G\epsilon_i = c^2\theta n_i \tag{13}$$

$$G\xi - \rho_0 a_0^2\lambda_i n_i = (\gamma-1)(\epsilon_i n_i - 3G\theta) \tag{14}$$

where the subscript 0 denotes a value at the wave head; n_i is the unit normal vector; and $u_n = u_i n_i$, $\lambda_i = [u_{i,j}]n_j$, $\epsilon_i = [q_{i,j}]n_j$, $\eta_i = [H_{i,j}]n_j$, $\xi = [p_{,i}]n_i$, $\zeta = [\rho_{,i}]n_i$ and $\theta = [p_{R,i}]n_i$ are defined on Σ.

For an advancing wave surface, Eqs. (9-14), on using the fact that $G \neq 0$, yield

$$G = c/3^{1/2} \tag{15}$$

$$G = C_{\mathrm{eff}_0} \tag{16}$$